Numerical Integration

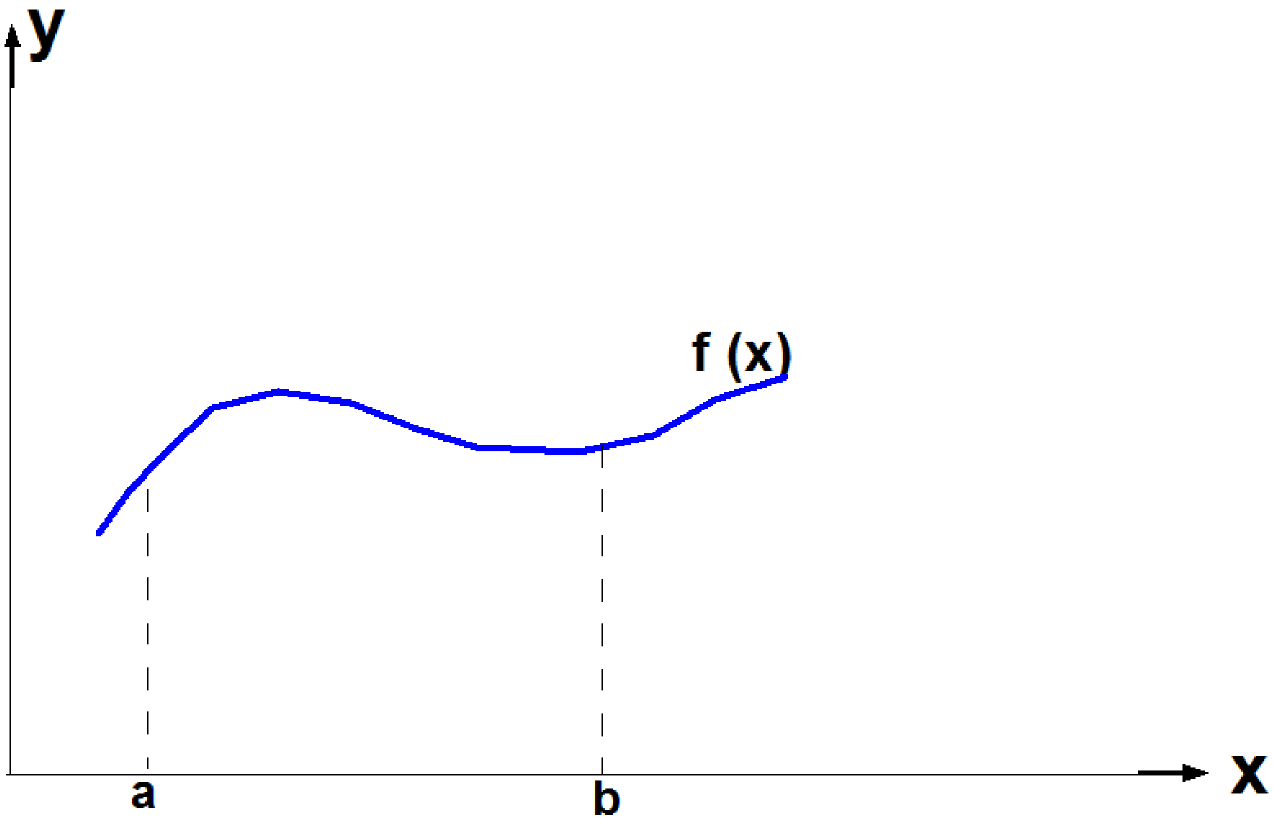
$$\int \frac{\sin x}{x} dx; \quad \int f(x)dx \text{ is not always calculated}$$

by direct integration, or $f(x)$ *is given in the form of a table;*

$$(x_o, y_o), (x_1, y_1), (x_2, y_2), \ldots\ldots\ldots\ldots (x_n, y_n)$$

$$\int_a^b f(x)\ dx \text{ is the area under the curve}$$

$$\text{from } x = a \text{ to } x = b$$



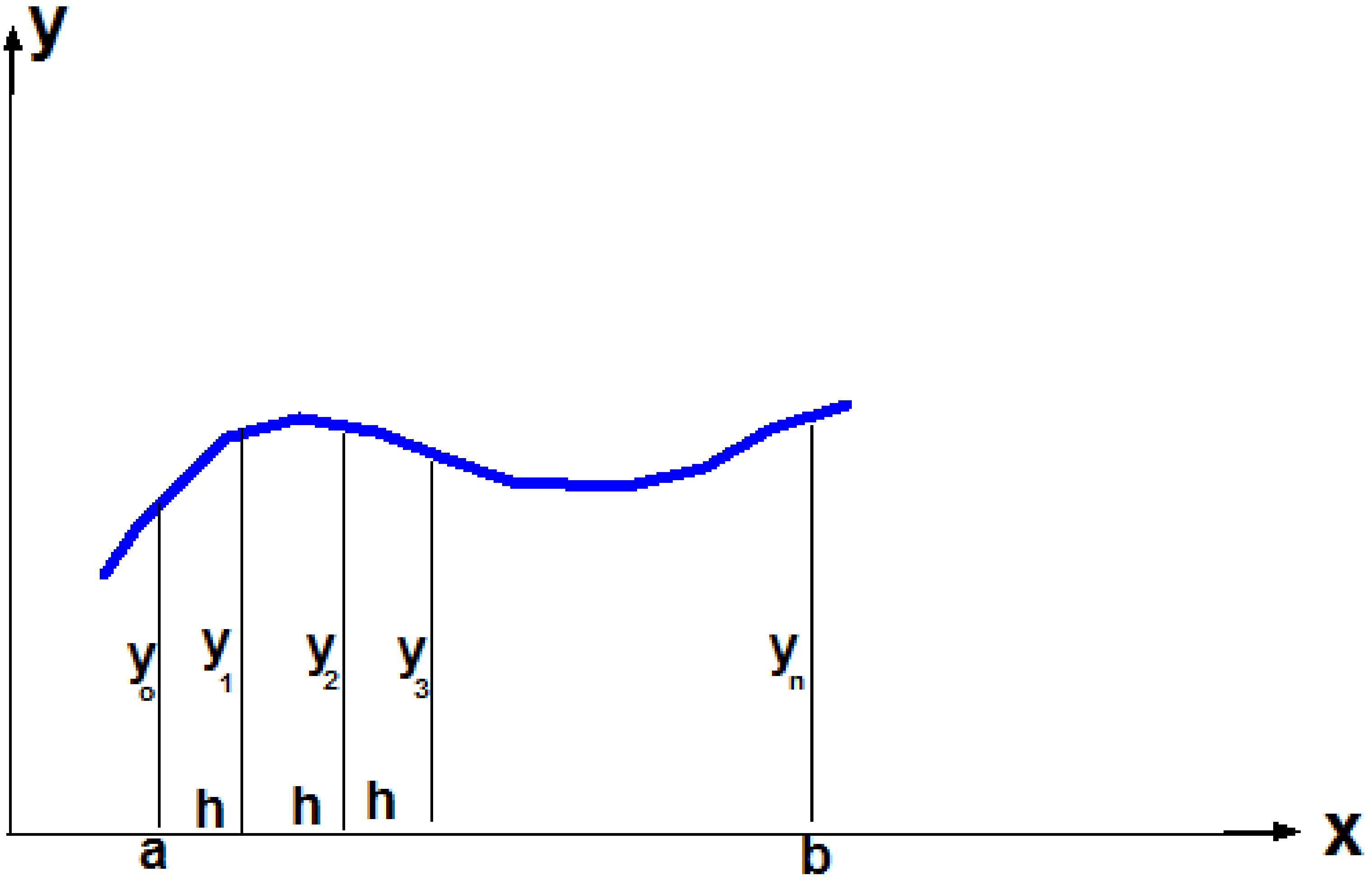
$$Methods\ of\ Calculated\ the\ area\ under\ the\ curve:$$

$$*Divide\ the\ distance\ (b-a)\ into\ (n)\ steps,$$

$$the\ length\ of\ each\ step\ is\ (h);$$

$$h = \frac{b-a}{n}$$

$$*Draw\ coordinates\ (y_o,\ y_1,\ y_2,\ldots\ldots\ldots y_n)$$


y
y
o
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1
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2
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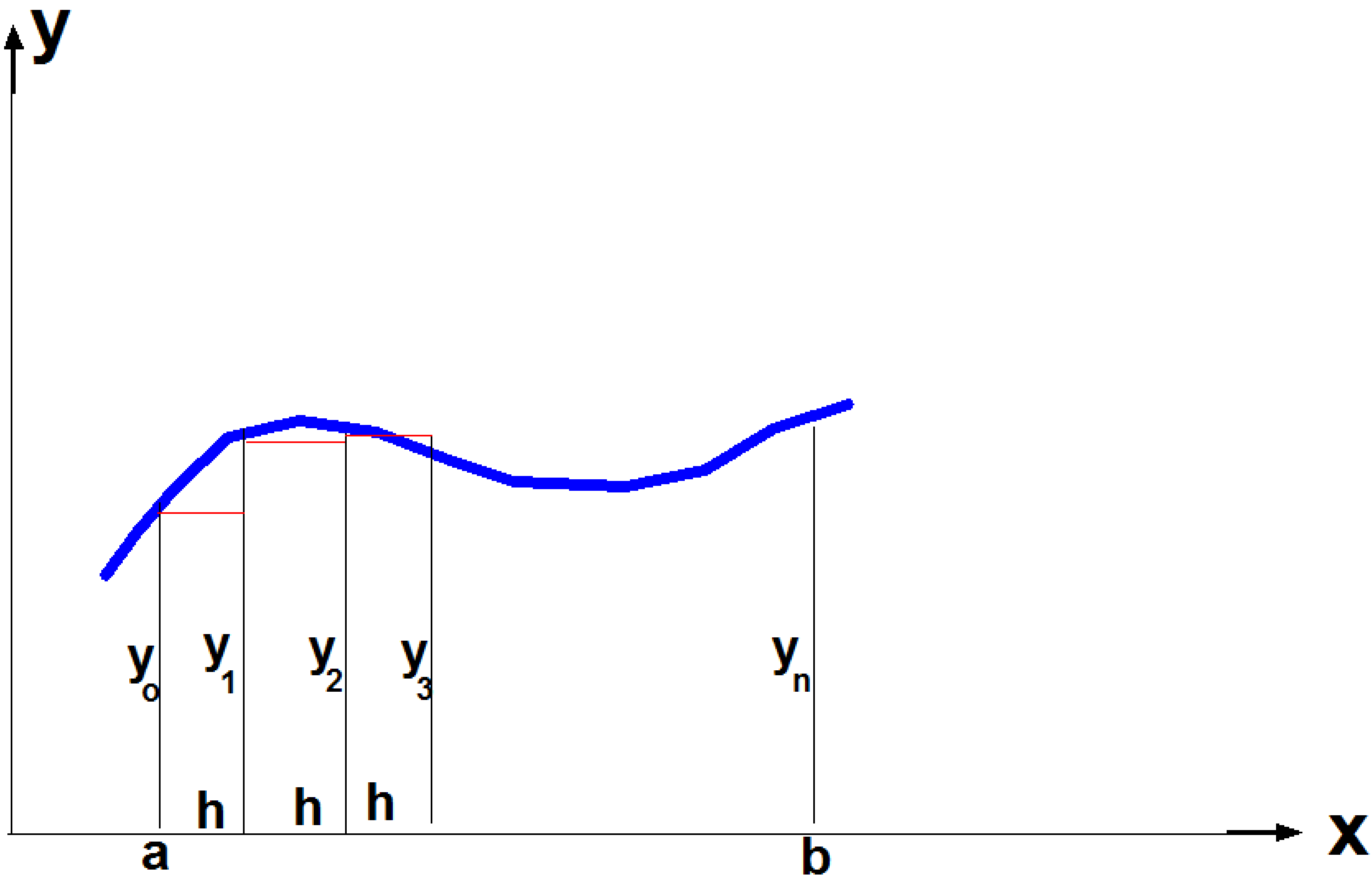
3
y
n
h
h
h
a
b
X

$The\ \ methods\ \ are:$

$1-The\ \ \ method\ \ of\ \ \ rec\tan gulars:$

$$I=\int_a^b f(x)\ \ dx$$

$$=h[y_o+y_1+y_2+\ldots\ldots\ldots\ldots\ldots y_{n-1}]$$


y
x
y o
y 1
y 2
y 3
y n
h
h
h
a
b

$2 - The\ \ method\ \ of\ \ trapizoids:$

$$I = \int_a^b f(x)\ dx$$

$$I = A_1 + A_2 + ...$$

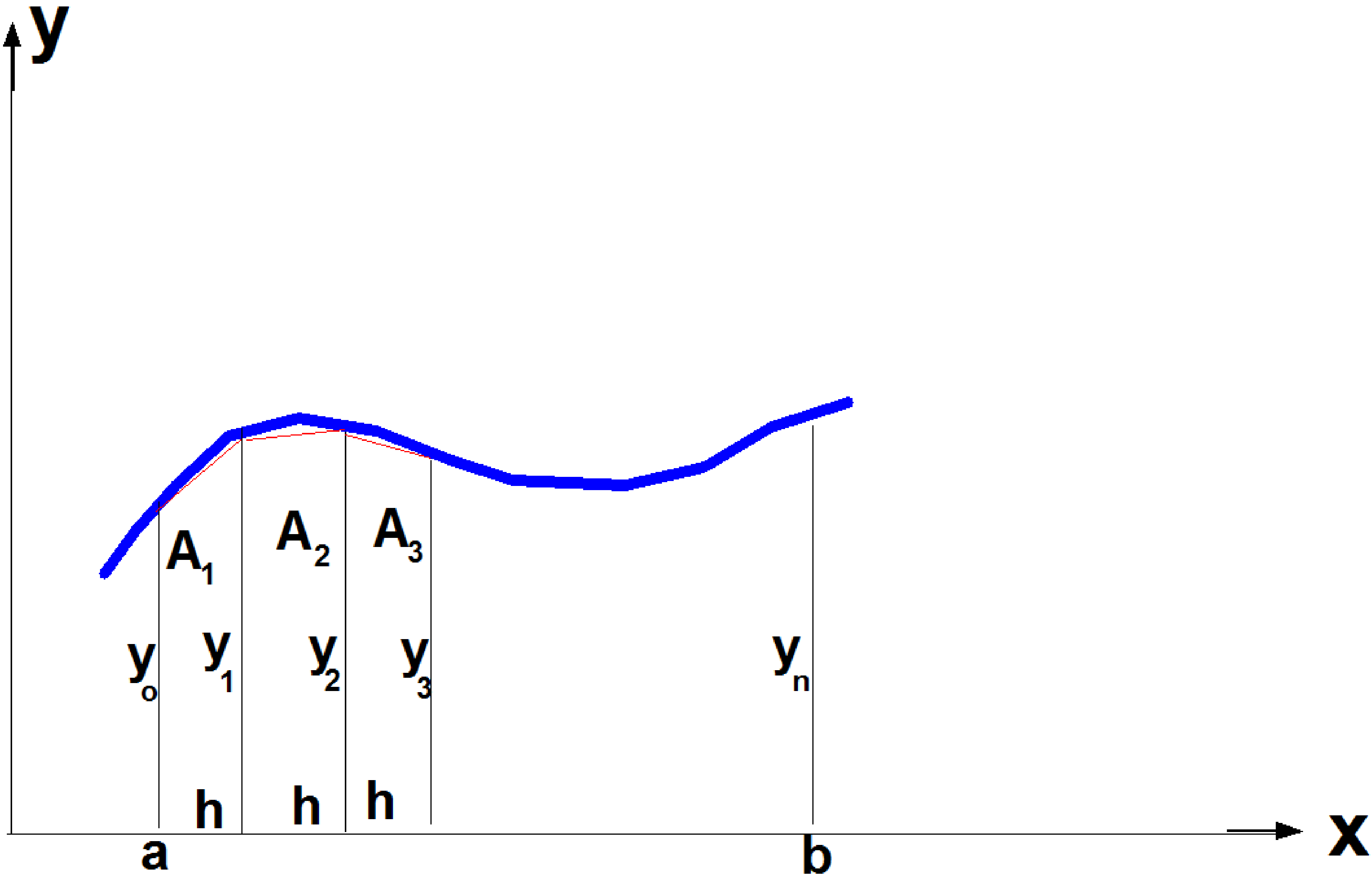
$$A_1 = h * \frac{y_o + y_1}{2}; \quad A_2 = h * \frac{y_1 + y_2}{2}; A_3 = h * \frac{y_2 + y_3}{2}$$

$$\vdots$$

$$A_n = h * \frac{y_{n-1} + y_n}{2}$$

$$\therefore I = \frac{h}{2}[y_o + 2y_1 + 2y_2 + 2y_3 + ...y_n]$$

$$= \frac{h}{2}[y_o + 2(y_1 + y_2 + y_3 + ...) + y_n]$$


y
A1
A2
A3
yo
y1
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h
h
h
a
b
x

$3-$ *Simpson's Rule:*

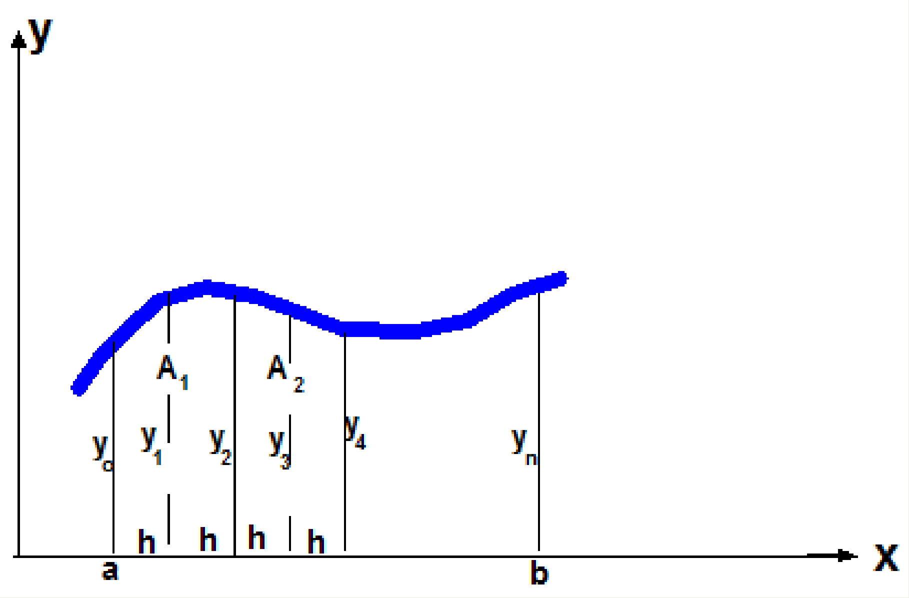
The quadratic equation is assumed to pass through the points. Thus the area into even number points.

$$A_1 = \frac{h}{3}[y_o + 4y_1 + y_2]$$

$$A_2 = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$A_3 = \frac{h}{3}[y_4 + 4y_5 + y_6]$$

$$\vdots$$

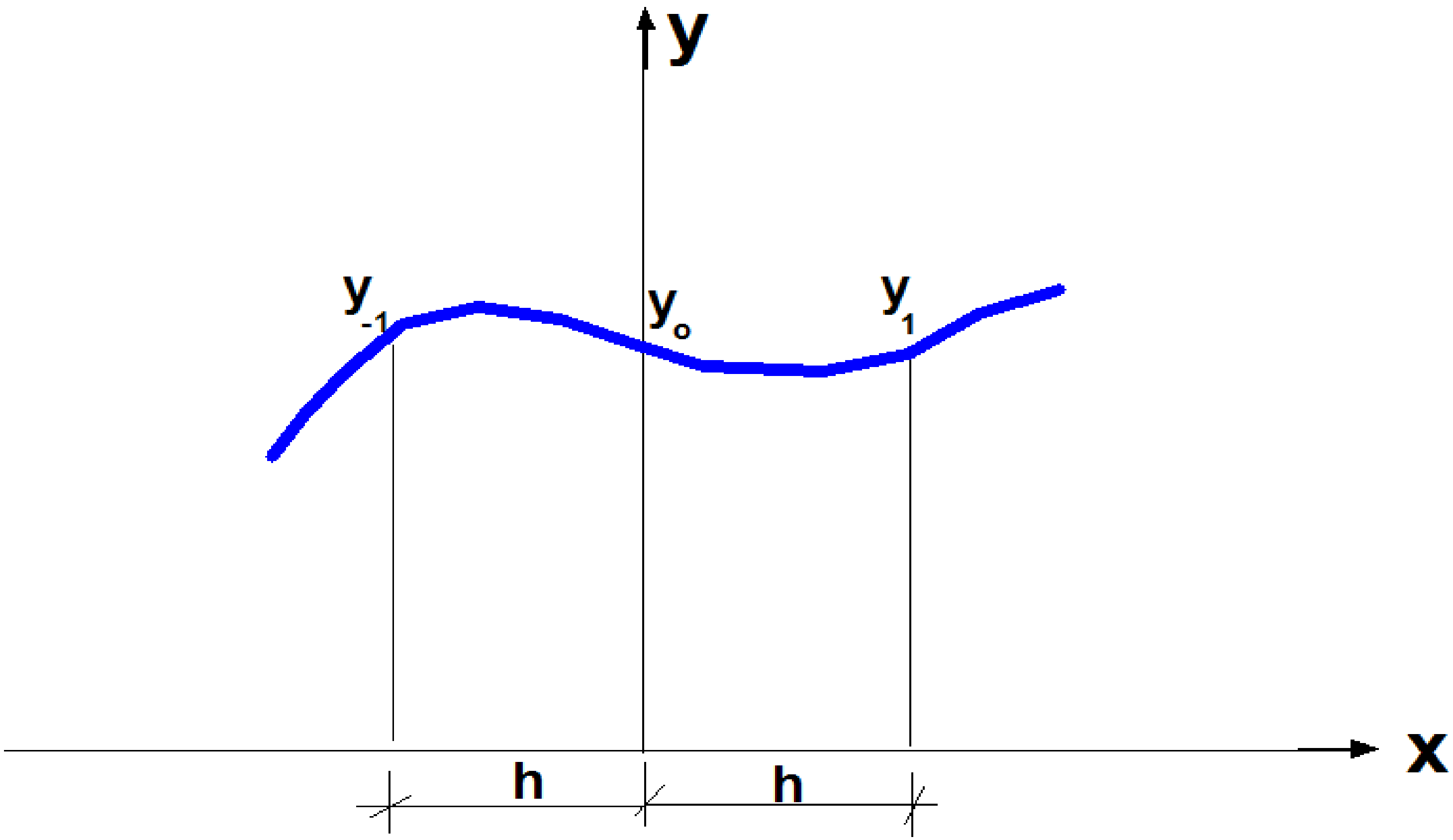


$$A = I = A_1 + A_2 + A_3 + \cdots$$

$$= \frac{h}{3}[y_o + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots) + y_n]$$

$$Assume\ a\ curve\ of\ \deg ree\ (2),\ Quadratic\ equation\ pass\ through\ three\ po\text{int}s:$$

$$[(-h, y_{-1}),\ (0, y_o)\ \&\ (h, y_1)].$$



$$y = a_o + a_1 x + a_2 x^2$$

$$At\ \ x = 0 \Rightarrow\ y_o = a_o + a_1(0) + a_2(0)^2$$

$$\therefore a_o = y_o$$

$$(-h, y_{-1})\ \ \Rightarrow\ y_{-1} = a_o + a_1(-h) + a_2(-h)^2\ \ \ldots\ldots\quad (1)$$

$$(h, y_1)\qquad \Rightarrow\ y_1 = a_o + a_1(h) + a_2(h)^2\ \ \ldots\ldots\quad (2)$$

$$y_{-1} = y_o - ha_1 + h^2 a_2$$

$$y_1 = y_o + ha_1 + h^2 a_2$$

Solve the above equations getting:

$$a_1 = \frac{y_1 - y_{-1}}{2h} \quad and \quad a_2 = \frac{y_{-1} - 2y_o + y_1}{2h^2}$$

$$\therefore y = y_o + \frac{y_1 - y_{-1}}{2h} x + \frac{y_{-1} - 2y_o + y_1}{2h^2} x^2$$

$$I = \int_{-h}^{h} [y_o + \frac{y_1 - y_{-1}}{2h} x + \frac{y_{-1} - 2y_o + y_1}{2h^2} x^2] dx$$

After integration and substitution get:

$$A = I = \frac{h}{3}[y_{-1} + 4y_o + y_1]$$

$$\underline{\underline{Example}}(1): Evaluate \int_0^1 \frac{\sin x}{x} dx \quad use\ n = 4\ panels.$$

$$\underline{\underline{Solution}}: \quad h = \frac{1-0}{4} = 0.25$$

x:	0	0.25	0.5	0.75	1.0
$f(x)$:	1	0.99	0.96	0.91	0.84

$$(1)\ By\ \ rec\tan gular\ \ method:$$

$$I = 0.25[1 + 0.99 + 0.96 + 0.91] = 0.965$$

$$\underline{OR}:$$

$$I = 0.25[0.99 + 0.96 + 0.91 + 0.84] = 0.925$$

$$Avg. = 0.945\,(from\ \ center\ \ of\ \ po\operatorname{int}s)$$

$$(2)\ By\ \ trapezoidd:$$

$$I = \frac{0.25}{2}[1 + 2(0.99 + 0.96 + 0.91) + 0.84] = 0.945$$

(4) *By Simpson's rule*:

$$I = \frac{0.25}{3}[1 + 4(0.99 + 0.91) + 2(0.96) + 0.84] = 0.9467$$

(2) *By Taylor Series*:

$$I = \int_0^1 \frac{\sin x}{x} dx$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

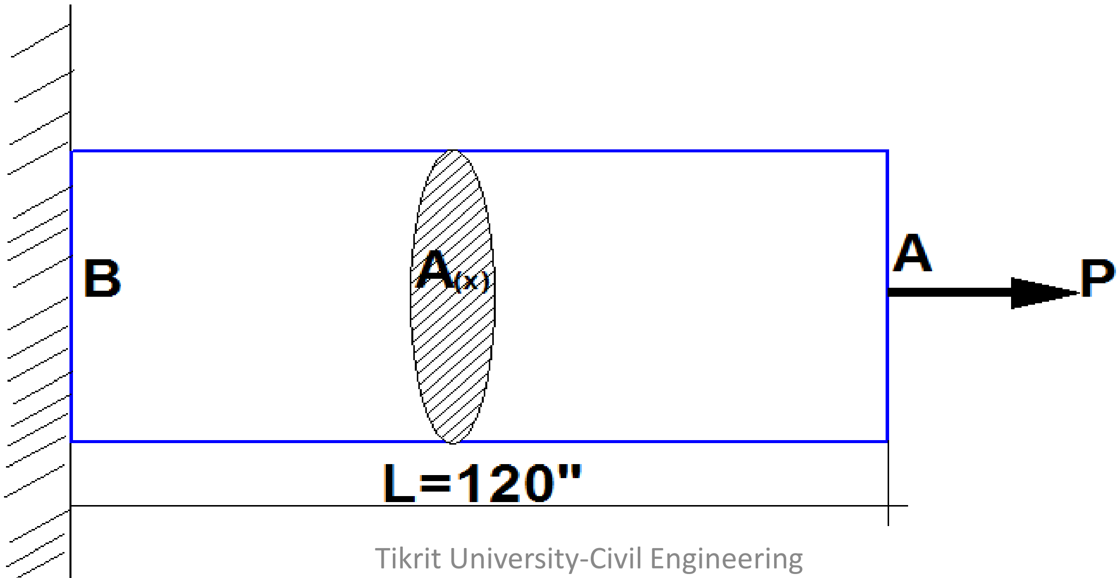
$$I = \int_0^1 [(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}) / x] dx$$

$$= \int_0^1 (1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}) dx = 0.946$$

Example (2): Determine the relative deflection of point (A) with respect to point (B).

$$P_{(x)} = 10 kips, \quad E = 30*10^3 ksi,$$

$$A = \left(-\frac{1}{3600}x^2 + \frac{1}{30}x + 1\right) in^2$$



Solution:

$$d\delta = \frac{P_{(x)}dx}{E\,A_{(x)}}$$

$$\delta = \int_0^{120} \frac{P_{(x)}dx}{E\,A_{(x)}} \quad , \; n = 6\;panels$$

$$h = \frac{120-0}{6} = 20$$

$$I = \delta = \int_0^{120} \frac{10 \quad dx}{30*10^3[-\frac{1}{3600}x^2 + \frac{1}{30}x+1]}$$

$$I = \int f(x)dx$$

$$f(x)=\frac{10}{30*10^{3}(-\frac{1}{3600}x^{2}+\frac{1}{30}x+1)}$$

$$f(x)=\frac{10*10^{-3}}{30(-\frac{1}{3600}x^{2}+\frac{1}{30}x+1)}$$

$$f(x)=10^{-4}*\frac{10}{3(-\frac{1}{3600}x^{2}+\frac{1}{30}x+1)}$$

$x:$	0	20	40	60	80	100	120
$f(x)=y*10^{-4}$:	3.33	2.143	1.765	1.667	1.765	2.143	3.33

$Simpson's\ \ Rule:$

$$S = I = \frac{10^{-4} * 20}{3}[3.33 + 4(2.143 + 1.667 + 2.147) + 2(1.765 + 1.765) + 3.33]$$

$\therefore I = 0.02502\,in$

Gauss – Quadrature formulas:

This formula is determined without specified pivot points that is given:

$$I = \int_A^B f(x)\,dx$$

The integral is approximated by a function of the form:

$$I = H_1 f(x_1) + H_2 f(x_2) + H_3 f(x_3) + \dots\dots\dots\dots + H_n f(x_n)$$

$$If\ there\ is\ one\ (\text{single})\ term\ [I = H_1 f(x_1)];$$

$$\int_A^B f(x)\,dx = H_1 f(x_1);\ two\ unknowns\ (H_1\ \&\ x_1)$$

$$So\ that\ the\ \text{int}egration\ must\ hold\ for:$$

$$f(x) = 1\ \&\ x_1;\ \int_A^B 1\,dx = H_1(1)$$

$$B - A = H_1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\int_A^B x\,dx = H_1\ x_1 \Rightarrow \frac{1}{2}(B^2 - A^2) = H_1 x_1 \ldots\ldots.(2)$$

$$Solve\ Eqs.\ (1)\,\&\,(2),\ getting;\ x_1 = \frac{B+A}{2}$$

$$\therefore I = H_1 f(x_1) = (B - A)\, f\left(\frac{B+A}{2}\right)$$

$Gauss - Quadrature\ formulas:$

Involving two and higher term result in sets of nonlinear algebraic equations. It's for this reason that the limit of integration is normalized as follows:

$$I = \int_A^B f(x)\,dx,\ Put\ x = c_o + c_1 r$$

$$x = A\,;\ r = -1 \Rightarrow \therefore A = c_o + c_1(-1)\ \ldots\ldots\ldots\ (1)$$

$$x = B\,;\ r = +1 \Rightarrow \therefore B = c_o + c_1(+1)\ \ldots\ldots\ldots(2)$$

$$c_o = \frac{1}{2}(A+B) \quad and \quad c_1 = \frac{1}{2}(B-A)$$

$$\therefore \; x = \frac{1}{2}(B+A) + \frac{1}{2}(B-A)\,r$$

$$dx = \frac{1}{2}(B-A)\; dr$$

$$I = \int_A^B f(x)\,dx$$

$$I = \int_{-1}^{+1} f[\frac{1}{2}(B+A) + \frac{1}{2}(B-A)\; r]\,[\frac{1}{2}(B-A)\,dr]$$

$$= \frac{1}{2}(B-A)\int_{-1}^{+1} f[\frac{1}{2}(B+A) + \frac{1}{2}(B-A)\; r]\,dr$$

$Two\ \ terms:$

$$I = H_1 f(r_1) + H_2 f(r_2)$$

$$f(r) = 1;\ r;\ r^2$$

$$\int_{-1}^{+1} 1\, dr = H_1(1) + H_2(1) = 2 \quad .... \quad (1)$$

$$\int_{-1}^{+1} r\ dr = H_1(-a) + H_2(a) = 0 \quad .... \quad (2)$$

$$\int_{-1}^{+1} r^2\ dr = H_1(-a)^2 + H_2(a)^2 = \frac{2}{3} \quad ..... \quad (3)$$

$Solve\ the\ above\ Eqs.,\ getting$

$$H_1 = H_2 = 1 \quad and \quad a = \pm\frac{1}{\sqrt{3}}$$

$$\therefore \int_{-1}^{1} f(r)dr = (1)\, f\left(\frac{-1}{\sqrt{3}}\right) + (1)\, f\left(\frac{1}{\sqrt{3}}\right)$$

$$Example\,(1):$$

$$\int_{-1}^{+1} x^2\,dx \quad two\ terms$$

$$Solution:$$

$$f(\frac{1}{\sqrt{3}}) = (\frac{1}{\sqrt{3}})^2 = \frac{1}{3}$$

$$f(\frac{-1}{\sqrt{3}}) = (\frac{-1}{\sqrt{3}})^2 = \frac{1}{3}$$

$$I = 1 * \frac{1}{3} + 1 * \frac{1}{3} = \frac{2}{3}$$

$$Three\ terms:$$

$$H_1 = \frac{5}{9} \Leftrightarrow a = -\sqrt{\frac{3}{5}}$$

$$H_2 = \frac{8}{9} \Leftrightarrow a = 0$$

$$H_3 = \frac{5}{9} \Leftrightarrow a = \sqrt{\frac{3}{5}}$$

$$Example\,(2):$$

$$\int_0^4 e^x\,dx \qquad three\,terms$$

$$Solution:$$

$$a = -\sqrt{3/5} = -0.774 \qquad H_1 = 0.555$$

$$a = 0 \qquad H_2 = 0.888$$

$$a = \sqrt{3/5} = 0.774 \qquad H_3 = 0.555$$

$$x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A)\ (r) \qquad \underline{\underline{or\ (a)}}$$

$$x_1(r=-0.774)=\frac{1}{2}(4+0)+\frac{1}{2}(4-0)(-0.774)=0.452$$

$$x_2(r=0)=\frac{1}{2}(4+0)+\frac{1}{2}(4-0)(0)=2$$

$$x_3(r=0.774)=\frac{1}{2}(4+0)+\frac{1}{2}(4-0)(0.774)=3.548$$

$$f(x_1) = e^{0.452} = 1.57145$$

$$f(x_2) = e^{2} = 7.38906$$

$$f(x_3) = e^{3.548} = 34.7438$$

$$I = \frac{1}{2}(4-0)[0.555*1.57145 + 0.888*7.38906$$

$$+ 0.555*34.7438]$$

$$\therefore I = 53.4463$$

$$Example: Evaluate \left[\int_0^1 \left(\sin x / x \right) dx \right] \quad use:$$

$$Gauss-Quadrature\ formula\ (n=3)$$

$Solution:$

$a = 0.774$ $\qquad H_1 = 0.555$

$a = 0$ $\qquad H_2 = 0.888$

$a = -0.774$ $\qquad H_3 = 0.555$

$$x = \frac{1}{2}(B+A) + \frac{1}{2}(B-A)\ (a) \quad \underline{\underline{or\ (r)}}$$

$$x_1\,(r=0.774)=\frac{1}{2}(1+0)+\frac{1}{2}(1-0)(0.774)=0.8873$$

$$x_2\,(r=0)=\frac{1}{2}(1+0)+\frac{1}{2}(1-0)(0)=0.5$$

$$x_3\,(r=-0.774)=\frac{1}{2}(1+0)+\frac{1}{2}(1-0)(-0.774)=0.1127$$

$$f(x_1) = \sin(0.8873) \big/ 0.8873 = 0.87385$$

$$f(x_2) = \sin(0.5) \big/ 0.5 = 0.95885$$

$$f(x_3) = \sin(0.1127) \big/ 0.1127 = 0.99788$$

$$I = \frac{1}{2}(1-0)[0.555 * 0.87385 + 0.888 * 0.95885$$

$$+ 0.555 * 0.99788]$$

$$\therefore I = 0.9451$$

$$Example: \int_1^3 e^x\,dx \qquad (use\ n = 4)$$

$$Solution:$$

$$a = \pm 0.861136 \quad , \quad \pm 0.339981$$

$$H = 0.347854 \quad , \quad 0.652145$$

$$x = \frac{1}{2}(B + A) + \frac{1}{2}(B - A)\ (a) \quad \underline{\underline{or\ (r)}}$$

$$x_1 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(0.861136) = 2.861136$$

$$x_2 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(-0.861136) = 1.138864$$

$$x_3 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(0.339981) = 2.339981$$

$$x_4 = \frac{1}{2}(3+1) + \frac{1}{2}(3-1)(-0.339981) = 1.660019$$

$$I=\frac{1}{2}(B-A)[H_1 f(x_1)+H_2 f(x_2)+H_3 f(x_3)+H_4 f(x_4)]$$

$$f(x_1)=e^{2.861136}=17.4813745$$

$$f(x_2)=e^{1.138864}=3.123218373$$

$$f(x_3)=e^{2.339981}=10.38103932$$

$$f(x_4)=e^{1.660019}=5.259410772$$

$$I=\frac{1}{2}(3-1)[0.347854*17.4813745+0.347854*3.123218373$$

$$+0.652145*10.38103932+0.652145*5.259410772]$$

$$\therefore I=17.36673$$